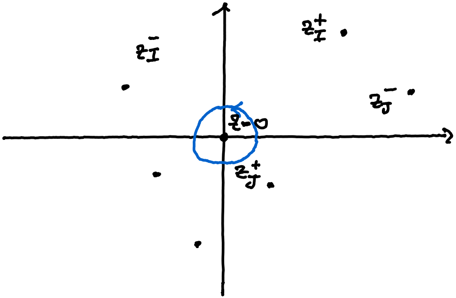
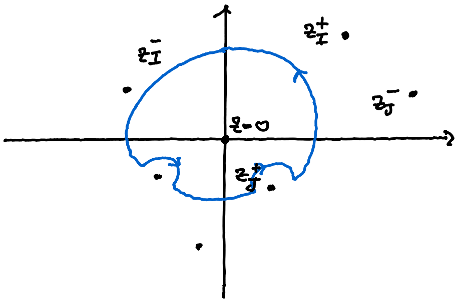
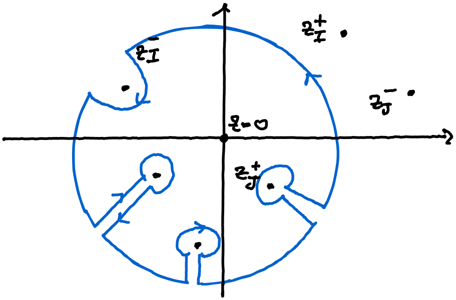
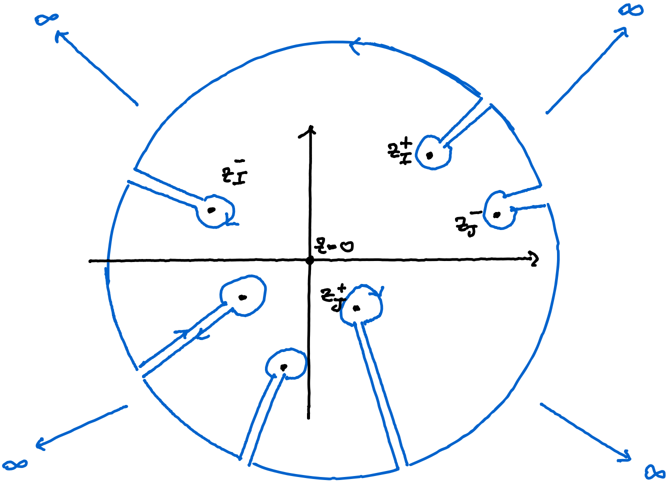


z_I^-
z_I^+
z_J^-
$z = 0$
z_J^+

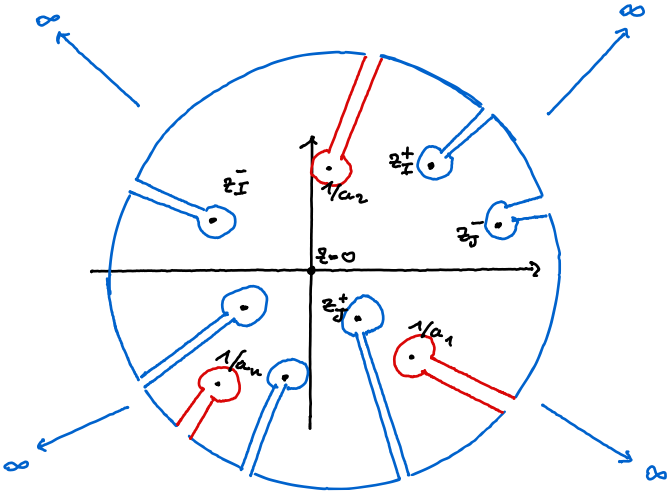

z_I^-
z_I^+
z_J^-
$z=0$
z_J^+



z_I^+
z_I^-
z_J^-
$z=0$
z_J^+



∞
∞
∞
∞
z_I^+
z_I^-
z_J^-
$z=0$
z_J^+


∞
∞
∞
∞
z_I^-
$1/a_2$
z_I^+
z_J^-
$z=0$
z_J^+
$1/a_1$
$1/a_n$

Part 2: Recursion Relations for One-Loop Goldstone Boson Amplitudes

Christoph Bartsch

Charles University, Prague

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IPNP Theory Friday

Based on: [2206.04694] CB, Karol Kampf, Jaroslav Trnka

One-loop amplitudes and integrands in the NLSM

- Similar to tree-level, can define an ordered amplitude at 1-loop

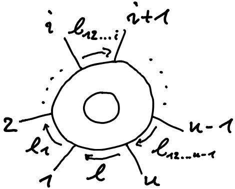
$$\mathcal{A}_n^{1-\text{loop}} = N \sum_{\sigma \in S_{n-1}} \frac{\langle t^{a_1} t^{a_{\sigma(2)}} \dots t^{a_{\sigma(n)}} \rangle}{(2F^2)^{n/2}} A_n^{1-\text{loop}}(p_1, p_{\sigma(2)}, \dots, p_{\sigma(n)}) + \dots,$$

  where "+..." now denotes terms suppressed by at least N^{-1}.
- Define corresponding *planar integrand* $\mathcal{I}_n^{1-\text{loop}}$:

$$A_n^{1-\text{loop}}(p_1, \dots, p_n) = \int d^4\ell \; \mathcal{I}_n^{1-\text{loop}}(\ell, p_1, \dots, p_n)$$

- For planar integrand can choose preferred loop-momentum routing (def. $\ell_{1\dots i} = \ell + p_1 + \dots + p_i$):

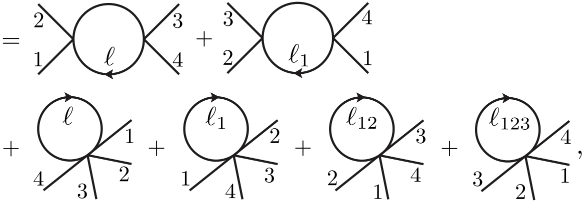
$$\mathcal{I}_n^{1-\text{loop}}(\ell, p_1, \dots, p_n) =$$



One-loop integrand at four points

- What can we say about $\mathcal{I}_n^{1-\text{loop}}$? Is it soft limit constructible? ($\Leftrightarrow$ Adler zero + consistent factorization)
- Let's have a look for $n = 4$. Compute Feynman diagrams:

$\mathcal{I}_4^{\text{FD}} =$



$$= 16u_3^2 - 32u_5 + \frac{n_4^{\text{FD}}}{\ell^2} + \frac{n_{42}}{\ell^2 \ell_{12}^2} + \text{cyc.}$$

- Numerators are: $n_4^{\text{FD}} = (4u_3 - 1)(2s_{12} + \ell_1^2 + \ell_{123}^2) + (8u_3 - 1)s_{23}$,

$$n_{42} = \tfrac{1}{2}(s_{12} + \ell_1^2)(s_{12} + \ell_{123}^2).$$

What about the Adler zero?

- Terms that involve coefficients u_3,u_5 integrate to zero in dim. reg..
- Idea: use this freedom in u_3,u_5 to impose Adler zero on ext. lines:

$$\lim_{p_k \to 0} \mathcal{I}_4^{1-\text{loop}}(\ell, p_j) \sim p_k \stackrel{!}{=} 0,$$

  $\rightarrow$ Impossible! Feynman integrand computed from $\mathcal{L}_2$ cannot satisfy Adler zero.

- Make a more general Ansatz:

$$\mathcal{I}_4^{\text{ans}}(\alpha_i) = \frac{\alpha_0}{4} + \frac{n_4^{\text{ans}}}{\ell^2} + \frac{n_{42}}{\ell^2 \ell_{12}^2} + \text{cyc.}$$

  with $n_4^{\text{ans}} = \alpha_1 s_{12} + \alpha_2 s_{23} + \alpha_3 \ell_1^2 + \alpha_4 \ell_{12}^2 + \alpha_5 \ell_{123}^2$, α_i free.

- Demanding Adler zero as above now gives a solution:

$$\alpha_0 = 2,\ \alpha_3 = -1,\ \alpha_4 = 1,\ \alpha_5 = -1.$$

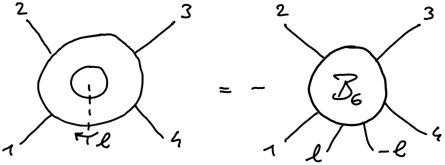
- This defines a two-parameter *soft integrand* $\mathcal{I}_4^{\text{S}} = \mathcal{I}_4^{\text{S}}(\alpha_1, \alpha_2)$.

What about consistent factorization?

- $\mathcal{I}_4^{\text{ans}}$ does not have any tree-level poles at $s_{i,i+1} = 0$. Residue is proportional to vanishing three-point tree amplitude.
- Consider loop-level poles $\ell_{1\dots i}^2 = 0$. Compute the cut at $\ell^2 = 0$:

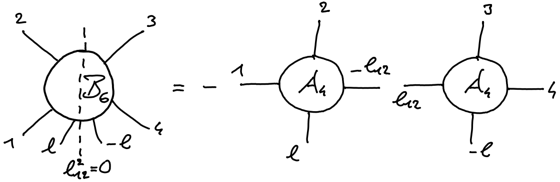
$$\begin{aligned}\text{Cut}[\mathcal{I}_4^{\text{S}}]_{\ell^2=0} &\equiv -B_6(p_1, p_2, p_3, p_4, -\ell, \ell) \\ &= \alpha_1 s_{12} + \alpha_2 s_{23} - \ell_1^2 + \ell_{12}^2 - \ell_{123}^2 + \frac{2n_{42}}{\ell_{12}^2}.\end{aligned}$$

- In $\mathcal{N} = 4$ sYM this cut corresponds to the forward limit of the 6-point tree-level amplitude. This is not the case in the NLSM.
But: B_6 is still an on-shell function with tree-level structure.



More on the B-function B_6

- B_6 is specified (up to terms $\sim \alpha_1$, α_2) by factorization on $\ell_{12}^2 = 0$:



and soft limits:

$$\lim_{p_2 \to 0} B_6 = \lim_{p_3 \to 0} B_6 = 0.$$

The latter are necessary conditions implied by the Adler zero of $\mathcal{I}_4^{\mathrm{S}}$.

- Furthermore, B_6 can be *fixed uniquely* by demanding

$$\lim_{\ell \to 0} B_6 = -(\alpha_1 + 2)s_{12} - \alpha_2 s_{23} \stackrel{!}{=} 0 \quad \Rightarrow \quad \alpha_1 = -2, \alpha_2 = 0,$$

in turn implying a *unique* soft integrand $\mathcal{I}_4^{\mathrm{S}}(\alpha_1\!=\!-2, \alpha_2\!=\!0)$.

Stepping back and enjoying the view

- Give explicit expressions for B_6:

$$B_6 = 2s_{12} + \ell_1^2 - \ell_{12}^2 + \ell_{123}^2 - \frac{(s_{12} + \ell_1^2)(s_{12} + \ell_{123}^2)}{\ell_{12}^2},$$

and the corresponding soft integrand $\mathcal{I}_4^{\mathrm{S}}$:

$$\mathcal{I}_4^{\mathrm{S}} = \frac{1}{2} - \frac{2s_{12} + \ell_1^2 - \ell_{12}^2 + \ell_{123}^2}{\ell^2} + \frac{1}{2}\frac{(s_{12} + \ell_1^2)(s_{12} + \ell_{123}^2)}{\ell^2 \ell_{12}^2} + \text{cyc.}$$

- At $n = 4$ points we have succeeded in finding an integrand that satisfies **(I)** Adler zero and **(II)** consistent factorization.
- Does that mean $\mathcal{I}_4^{\mathrm{S}}$ is soft limit constructible? $\rightarrow$ Yes!
- But wait, there's more: B_6 is *also* soft limit constructible from tree-level factorization and soft limits.

Recursion for B_6

- For the recursion of B_6 we use shift ($q_i^2 = p_i \cdot q_i = 0,\ i = 1, 4$):

$$\hat{p}_1(z) = p_1 + zq_1, \quad \hat{p}_2(z) = (1 - a_2 z)p_2,$$
$$\hat{p}_3(z) = (1 - a_3 z)p_3, \quad \hat{p}_4(z) = p_4 + zq_4.$$

  Additionally we shift $\hat{\ell}(z) = (1 - bz)\ell$.

- Then evaluate (using $F_{B,6}(z) = (1 - a_2 z)(1 - a_3 z)(1 - bz)$)

$$B_6(z{=}0) = \oint_{z=0} \frac{dz}{z} \frac{B_6(z)}{F_{B,6}(z)} = \ldots$$
$$= -\sum_{z_i} \operatorname*{Res}_{z=z_i} \left(\frac{A_4^L(z) A_4^R(z)}{z \hat{\ell}_{12}^2(z) F_{B,6}(z)} \right) - \frac{A_4^L(0) A_4^R(0)}{\ell_{12}^2},$$

  where $z_i \in \{a_2^{-1}, a_3^{-1}, b^{-1}\}$.

Recursion for $\mathcal{I}_4^{\mathrm{S}}$

- Use the same shift $\hat{p}_i(z)$ as for B_6. In addition, shift $\hat{\ell}(z) = l + zq$, with $q^2 \neq 0 \neq l \cdot q$.
- Then evaluate (using $F_{\mathcal{I},4}(z) = (1 - a_2 z)(1 - a_3 z)$)

$$\mathcal{I}_4^{\mathrm{S}}(z\!=\!0) = \oint_{z=0} \frac{dz}{z} \frac{\mathcal{I}_4^{\mathrm{S}}(z)}{F_{\mathcal{I},4}(z)} = \sum_{m=0}^{3} \sum_{\pm} \operatorname*{Res}_{z=z_m^{\pm}} \left(\frac{B_6(z)}{z\, \ell^2_{1\ldots m}(z) F_{\mathcal{I},4}(z)} \right),$$

  where $z_m^{\pm}$ are the solutions to the quadratic equation $\hat{\ell}^2_{1\ldots m}(z_m^{\pm}) = 0$.
- This concludes our story at four points.

  $\rightarrow$ What can we say for *higher points*?

The B-function B_{n+2}

- At n points $B_{n+2}(p_1, \ldots, p_n, -\ell, \ell)$ corresponds to the single cut ($\ell^2 = 0$) of the soft integrand $\mathcal{I}_n^{\mathrm{S}}$.
- It is characterized as the *unique* function satisfying **(II)** consistent factorization on poles $\ell_{1\ldots m}^2 = 0$,

$$B_{n+2} \xrightarrow{\ell_{1\ldots m}^2 \to 0} -\frac{A_{n_L}(\ell, p_1, \ldots, p_m, -\ell_{1\ldots m}) A_{n_R}(\ell_{1\ldots m}, p_{m+1}, \ldots, p_n, -\ell)}{\ell_{12\ldots m}^2},$$

where $n_L = m + 2$, $n_R = n - m + 2$ for $m = 2, 4, \ldots, n-2$. Residues at $s_{i,i+m} = 0$ are a bit more involved but can also be prescribed.
At the same time B_{n+2} obeys **(I)** the Adler zero ($i = 2, 3, \ldots, n-1$):

$$\lim_{p_i \to 0} B_{n+2} = 0, \qquad \lim_{\ell \to 0} B_{n+2} = 0.$$

- B_{n+2} is soft limit constructible from tree-level amplitudes and lower-point B-functions. This has been verified up to B_{10}.

The soft integrand $\mathcal{I}_n^{\mathrm{S}}$

- The soft integrand $\mathcal{I}_n^{\mathrm{S}}$ has single cuts ($m = 0, \ldots, n-1$):

$$\mathcal{I}_n^{\mathrm{S}} \xrightarrow{\ell_{1\ldots m}^2 \to 0} -\frac{B_{n+2}(p_{i+1}, \ldots, p_n, p_1, \ldots, p_i, -\ell_{1\ldots m}, \ell_{1\ldots m})}{\ell_{1\ldots m}^2}.$$

- On tree-level poles $\mathcal{I}_n^{\mathrm{S}}$ factorizes (schematically) as ($i = 1, \ldots n$)

$$\mathcal{I}_n^{\mathrm{S}} \xrightarrow{s_{i,i+m} \to 0} -\frac{A_{n_L} \mathcal{I}_{n_R}^{\mathrm{S}} + \mathcal{I}_{n_L}^{\mathrm{S}} A_{n_R}}{s_{i,i+m}},$$

  where now $n_L = m + 2$, $n_R = n - m$, and $m = 2, 4, \ldots, n-4$.

- Finally, $\mathcal{I}_n^{\mathrm{S}}$ satisfies the Adler zero on all external lines

$$\lim_{p_i \to 0} \mathcal{I}_n^{1-\mathrm{loop}}(\ell, p_j) = 0.$$

- $\mathcal{I}_n^{\mathrm{S}}$ is soft limit constructible from the B-function B_{n+2} and lower point tree-amps and soft integrands. This has been verified up to $\mathcal{I}_8^{\mathrm{S}}$.

What's next?

- Can this construction be extended to two loops and beyond?
- Are there other exceptional EFTs for which a soft integrand can be found?
- Is there some (positive?) geometric interpretation for the soft integrand?
- Does there exist a Lagrangian that can directly compute the soft integrand?

Thank you for your attention!