Order parameters of three-flavour chiral symmetry from $\pi\pi$ scattering

WIP, to be published

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Short history of π-π scattering

Yukawa formulated his theory about pions as mediators of nuclear forces in 1934

They were discovered the next decade (1947 by Powell et al., he received the Nobel prize for it in 1950)

Nowadays - low energy amplitudes of π-π scattering are known with relatively good precision

The results from theory come from several sources:

- chiral perturbation theory (χPT) - Weinberg [1979], Gasser and Leutwyler [1985]
- Roy equations (model independent method based on dispersion relations) - Roy [1971]

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Standard Lagrangian density of QCD can be written as

$$\mathcal{L}_{QCD} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \overline{q} \left(i\gamma_\mu D^\mu - M \right) q \tag{1}$$

with invariants called *color* and *baryon number*.

Now looking at masses of quarks → construct an additional approximate symmetry SU(N_f), where N_f is the number of quarks we consider to be of the same mass.

Next we may assume the small masses → splits the symmetries to 2 - left and right helicities.

Effective Lagrangian

Introduced by S. Weinberg

Idea of expanding Lagrangian in degrees of freedom below a certain boundary.

Such expansion of QCD $\equiv$ *chiral perturbation theory*.

This approach has its difficulties - NLO gives us 10 coupling constants and NNLO terrifying 90.

QCD effective Lagrangian by Gasser and Leutwyler

$$\mathcal{L}_{eff} = \mathcal{L}^{(2)} + \mathcal{L}^{(4)} + \mathcal{L}^{(6)} + \ldots \tag{2}$$

$$\mathcal{L}^{(2)} = \frac{F_0^2}{4} Tr[D_\mu U D^\mu U^+ + \left(U^+ \chi + \chi^+ U\right)] \tag{3}$$

$$\mathcal{L}^{(4)} = \mathcal{L}^{(4)}(L_1, .., L_{10}) + \mathcal{L}^{(4)}_{WZ} \tag{4}$$

$$U(x) = e^{\frac{i}{F_0}\phi^a(x)\lambda^a} \tag{5}$$

$$\chi = 2B_0 M. \tag{6}$$

Effective Lagrangian

$$\begin{aligned}\mathcal{L}^{(4)}(L_1,..,L_{10}) \quad = \quad & L_1 Tr[D_\mu U^+ D^\mu U]^2 + L_2 Tr[D_\mu U^+ D_\nu U] Tr[D^\mu U^+ D^\nu U] + \\ & + L_3 Tr[D_\mu U^+ D^\mu U D_\nu U^+ D^\nu U] + \\ & + L_4 Tr[D_\mu U^+ D^\mu U] Tr[\chi^+ U + \chi U^+] + \\ & + L_5 Tr[D_\mu U^+ D^\mu U(\chi^+ U + U^+ \chi)] + L_6 Tr[\chi^+ U + \chi U^+]^2 + \\ & + L_7 Tr[\chi^+ U - \chi U^+]^2 + L_8 Tr[\chi^+ U \chi^+ U + \chi U^+ \chi U^+] - \\ & - iL_9 Tr[F_R^{\mu\nu} D_\mu U D_\nu U^+ + F_L^{\mu\nu} D_\mu U^+ D_\nu U] + \\ & + L_{10} Tr[U^+ F_R^{\mu\nu} U F^L_{\mu\nu}] \end{aligned} \tag{7}$$

$$\mathcal{L}^{(6)} = \mathcal{L}^{(6)}(C_1,..,C_{90}) + \mathcal{L}^{(6)}_{WZ}(C_1^W,..,C_{23}^W) \tag{8}$$

Resummed approach

Introduced by Stern et al.

Resummed approach aims to bypass this problem - it resumms the higher order terms and doesn't omit them.

The resummed approach yields on calculating this remainder directly, but assumes we may describe it as the observable itself multiplied by an unknown variable.

One may place restrictions on this remainder, when calculating the original observable.

So an observable with resummed approach may look i.e. like this

$$A = A_{LO} + A_{NLO} + A\delta A \tag{9}$$

Consider a good globally convergent observable A. Let's have a look at the convergence of $B \equiv \frac{1}{A}$. Using terms up to NLO

$$A = A_{LO} + A_{NLO} + A\delta A \tag{10}$$
$$B = B_{LO} + B_{NLO} + B\delta B, \tag{11}$$

we can express the first few terms of B as

$$B_{LO} = \frac{1}{A_{LO}} \tag{12}$$
$$B_{NLO} = -\frac{A_{NLO}}{A_{LO}^2} \tag{13}$$
$$\delta B = \frac{(1 - X_A)^2}{X_A^2} - \frac{\delta A}{X_A^2} \tag{14}$$

defining

$$X_A \equiv \frac{A_{LO}}{A} \tag{15}$$

χPT LO parameters X,Y and Z

The principal order parameters of QCD in spontaneous symmetry breaking of chiral symmetry are the quark condensate and pseudoscalar decay constants

$$\Sigma(N_f) = - < 0 \mid \overline{q}q|0 >|_{m_q \to 0} \tag{16}$$

$$F(N_f) = F_P^a \mid_{m_q \to 0} \tag{17}$$

$$ip_\mu F_P^a =< 0 \mid A_\mu^a \mid P >, \tag{18}$$

where N_f = number of light quarks with mass m_q, A_μ^a are axial vector currents and F_P^a are the decay constants of light pseudoscalar mesons P.
We can reparametrize these and relate them to physical quantities connected with pion two point Green functions

$$X(N_f) = \frac{2\hat{m}\Sigma(N_f)}{F_\pi^2 M_\pi^2} \tag{19}$$

$$Z(N_f) = \frac{F(N_f)^2}{F_\pi^2} \tag{20}$$

$$Y = \frac{X(N_f)}{Z(N_f)}, \tag{21}$$

with $\hat{m} = \frac{m_u + m_d}{2}$ and M_π the physical mass.

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In $\pi\pi$ scattering, we have

$$\pi_1(p_1)\pi_2(p_2) \longrightarrow \pi_3(p_3)\pi_4(p_4) \tag{22}$$

From QFT it follows that the amplitudes of the s, t and u channels should look like

$$A_{\pi\pi}(s,t,u) = B_{\pi\pi}(t,s,u) = C_{\pi\pi}(u,t,s) \tag{23}$$

Assuming unitarity of S-matrix

$$S^{+}S = 1, \tag{24}$$

we can introduce the transition matrix

$$iT = S - 1 \tag{25}$$

$$-i(T - T^{+}) = T^{+}T \tag{26}$$

and finally the amplitude A_{fi}

$$\langle f|T|i\rangle = (2\pi)^4 N_{P_f} N_{P_i} \delta^{(4)}(P_f - P_i) i A_{fi}, \tag{27}$$

where i, f are initial and final states, P_i, P_f sum of momenta in these states. N_P are the standard normalisation factors

$$N_p = \frac{1}{(2\pi)^{\frac{3}{2}}(2p_0)^{\frac{1}{2}}}. \tag{28}$$

Inserting intermediate states and assuming time-reversal invariance, we arrive at the so called "Cutkosky rule"

$$2\mathrm{Im}A_{fi} = \sum_n (2\pi)^4 N_{P_n} \delta^{(4)}(P_n - P_i) A^{*}_{nf} A_{ni}. \tag{29}$$

Dispersion relations → useful tool allowing us to relate higher order amplitudes to the amplitudes of the lower order

Consider a complex function $F(s)$ with s a complex argument, using these assumptions:

- $F(s)$ has a branch cut for real $s > M^2$.
- $F(s)$ is real for $s < M^2$.
- $F(s)$ is analytic for any complex s (except along the branch cut).

For some point s_0 inside the "Pac-Man" curve along the branch cut, we get

$$F(s_0) = P_n(s_0) + \frac{s_0^n}{\pi} \int_{M^2}^{\infty} \frac{\mathrm{Im} F(s)}{(s - s_0 - i\epsilon) s^n} ds, \tag{30}$$

with $P_n(s_0)$ a polynomial of (n-1)-th order.

In second order of QFT, the dispersion relations + analyticity + crossing symmetry + unitarity give us (for a fixed $u = u_0$)

$$\begin{aligned} A_{\pi\pi}(s,t,u_0) =& P^{(2)}(s,t,u_0) + \frac{s^3}{\pi}\int_{m_s^2}^{\infty} \frac{\mathrm{Im} A_{\pi\pi}(x,y,u_0)}{x-s}\frac{dx}{x^3} + \\ &+ \frac{t^3}{\pi}\int_{m_t^2}^{\infty} \frac{\mathrm{Im} B_{\pi\pi}(x,y,u_0)}{x-t}\frac{dx}{x^3}, \end{aligned} \tag{31}$$

with the polynom of at most second order in Mandelstam variables, defining $y = \sum_{i=1}^{4} m_i^2 - u_0 - x$.

Now we divide the integrals into low energy region and region above the threshold. Thanks to the reconstruction theorem the high energy part up to the $O\left(\left(\frac{p}{\Lambda}\right)^6\right)$ order can be absorbed into the subtraction polynomial, but of a higher order:

$$\begin{aligned} A_{\pi\pi}(s,t,u_0) =& P^{(3)}(s,t,u_0) + \frac{s^3}{\pi}\int_{m_s^2}^{\Lambda^2} \frac{\mathrm{Im} A_{\pi\pi}(x,y,u_0)}{x-s}\frac{dx}{x^3} \\ &+ \frac{t^3}{\pi}\int_{m_t^2}^{\Lambda^2} \frac{\mathrm{Im} B_{\pi\pi}(x,y,u_0)}{x-t}\frac{dx}{x^3}. \end{aligned} \tag{32}$$

Roy equations - dispersion relations with partial wave decomposition

Roy's representation for the partial wave amplitudes t_l^I of elastic $\pi\pi$ scattering reads

$$t_l^I(s) = k_l^I(s) + \sum_{I'=0}^{2} \sum_{l'=0}^{\infty} \int_{4M_\pi^2}^{\infty} ds' K_{ll'}^{II'}(s, s') \mathrm{Im} t_{l'}^{I'}(s'), \tag{33}$$

with I as isospin and l as angular momentum.

The $k_l^I(s)$ is the partial wave projection of the subtraction term (present only in s and p-waves).

Validity of these equations has been established on the interval $4M_\pi^2 < s < 60M_\pi^2$.

Now we may finally express the amplitude of $\pi\pi$ scattering up to $O(p^6)$ as

$$\begin{aligned} A_{\pi\pi}(s|t,u) =& \frac{\alpha_{\pi\pi}}{3F_\pi^2} M_\pi^2 + \frac{\beta_{\pi\pi}}{3F_\pi^2}(3s - 4M_\pi^2) + \\ &+ \frac{\lambda_1}{F_\pi^4}\left(s - 2M_\pi^2\right)^2 + \frac{\lambda_2}{F_\pi^4}\left[(t - 2M_\pi)^2 + (u - 2M_\pi)^2\right] + \\ &+ \frac{\lambda_3}{F_\pi^6}\left(s - 2M_\pi^2\right)^2 + \frac{\lambda_4}{F_\pi^6}\left[(t - 2M_\pi)^3 + (u - 2M_\pi)^3\right] + \\ &+ \bar{J}(s|t,u) + O\left[\left(\frac{p}{\Lambda_H}\right)^8\right], \end{aligned} \tag{34}$$

where $\alpha_{\pi\pi}$, $\beta_{\pi\pi}$, $\lambda_1 \ldots \lambda_4 \equiv$ *subthreshold parameters*

$\bar{J}(s|t,u)$ collects the unitary cuts arising from elastic $\pi\pi$ intermediate states.

We focus on the LO subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$.

Parametrization obviously isn't fixed $\rightarrow$ one may use other parameters instead of $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ (representation from S. Descotes-Genon, N. H. Fuchs, L. Girlanda and J. Stern), i.e. scattering lengths a_0^0 and a_0^2 ($\equiv$ "phenomenological representation" from S. M. Roy)

$$\begin{aligned} A_{\pi\pi}(s,t,u) =& 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2}\left(2a_0^0 - 5a_0^2\right)s + P(s,t,u) + 32\pi[\frac{1}{3}\overline{W}^0(s) + \\ & \frac{3}{2}(s-u)\overline{W}^1(t) + \frac{3}{2}(s-t)\overline{W}^1(u) + \frac{1}{2}\overline{W}^2(u) + \frac{1}{2}\overline{W}^2(u) - \\ & \frac{1}{3}\overline{W}^2(s)] + O(p^8), \end{aligned} \tag{35}$$

	$\alpha_{\pi\pi}$	$\beta_{\pi\pi}$
Descotes et al. [2002]	1.381 ± 0.242	1.081 ± 0.023
Colangelo et al. [2001]	1.08 ± 0.07	1.12 ± 0.01

Table: Comparison of values for the subthreshold parameters

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We're going to use Bayesian approach, which can be derived as

$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A) \tag{36}$$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}. \tag{37}$$

Using partition of one and generalizing for continuous sets, we get

$$P(x|y) = \frac{P(y|x)P(x)}{\int_\Omega P(y|x)P(x)dx} \tag{38}$$

For physical observables, we can use

$$P(data|true) = \prod_k \frac{1}{\sigma_k\sqrt{2\pi}} \exp\left[-\frac{(O_k^{exp} - O_k^{true})^2}{2\sigma_k^2}\right], \quad (39)$$

For a theory with parameters X_i, we may rewrite this into

$$P(data|X_i) = \prod_k \frac{1}{\sigma_k\sqrt{2\pi}} \exp\left[-\frac{(O_k^{exp} - O_k^{theory}(X_i))^2}{2\sigma_k^2}\right]. \quad (40)$$

$P(data|X_i)$ is the probability density of obtaining observed values of the observables O_k^{exp} with X_i as the true values of the parameters.

Consider an experiment of n measurements with the result

$$\mu = \overline{x} \pm \frac{\sigma}{\sqrt{n}}. \tag{41}$$

What we want is

$$P(\overline{x} - \frac{\sigma}{\sqrt{n}} \leq \mu \leq \overline{x} + \frac{\sigma}{\sqrt{n}}) = 68\%, \tag{42}$$

but conventional statistics only gives us

$$P(\mu - \frac{\sigma}{\sqrt{n}} \leq \overline{X} \leq \mu + \frac{\sigma}{\sqrt{n}}) = 68\%. \tag{43}$$

Bayesian approach - observables with correlation

Using an example of two variables, A and B with normal distributions, we may express probability of A and B as

$$
\begin{aligned}
P(A_{exp}, B_{exp}|A, B) &= \frac{\sqrt{C}}{2\pi} e^{-\frac{1}{2} V^T C V} && (44) \\
V &= \begin{pmatrix} A - A_{exp} \\ B - B_{exp} \end{pmatrix}, & C &= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \\
c_{11} &= \frac{1}{\sigma_A^2 \left(1 - \rho_{AB}^2\right)}, & c_{22} &= \frac{1}{\sigma_B^2 \left(1 - \rho_{AB}^2\right)} \\
c_{12} = c_{21} &= \frac{-\rho_{AB}}{\sigma_A \sigma_B \left(1 - \rho_{AB}^2\right)}.
\end{aligned}
$$

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We may calculate the subthreshold parameters as (working in SU(3) and identifying X(3)≡ X, Z(3)≡ Z)

$$\begin{aligned}\alpha_{\pi\pi} =& 1+\frac{3r}{r+2}\epsilon(r)-\frac{2Yr}{r+2}\eta(r)+\frac{2(1-X)}{r+2}+\frac{4(1-Y)}{r+2}-\frac{1}{2}Y^2\left(\frac{M_\pi}{4\pi F_\pi}\right)^2\cdot\\ &\cdot\left(\frac{r}{(r-1)(r+2)}\left((r+2)\log\left(\frac{M_\eta^2}{M_K^2}\right)-(r-2)\log\left(\frac{M_K^2}{M_\pi^2}\right)\right)+\frac{7}{3}\right)-\\ &-\frac{6}{r+2}\left(\frac{r+1}{r-1}\delta_{M_\pi}-\left(\epsilon(r)+\frac{2}{r-1}\right)\delta_{M_K}\right)-\\ &-Y\frac{2r}{r+2}\left(\frac{r+1}{r-1}\delta_{F_K}-\left(\eta(r)+\frac{2}{r-1}\right)\delta_{F_K}\right)+2Y\delta_{F_\pi}+\delta_{\alpha_{\pi\pi}}\end{aligned} \tag{45}$$

$$\begin{aligned}\beta_{\pi\pi} =& 1+\frac{r\eta(r)}{r+2}+\frac{2(1-Z)}{r+2}+\frac{1}{2}Y\left(\frac{M_\pi}{4\pi F_\pi}\right)^2\cdot\\ &\cdot\left(\frac{r}{(r-1)(r+2)}\left((2r+1)\log\left(\frac{M_\eta^2}{M_K^2}\right)+(4r+1)\log\left(\frac{M_K^2}{M_\pi^2}\right)\right)-5\right)-\\ &-\frac{2}{r+2}\left(\frac{r+1}{r-1}\delta_{F_\pi}-\left(\eta(r)+\frac{2}{r-1}\right)\delta_{F_K}\right)+\delta_{\beta_{\pi\pi}}\end{aligned} \tag{46}$$

where

$$Y = \frac{X}{Z} \tag{47}$$

$$r = \frac{m_s}{\hat{m}} \tag{48}$$

$$\epsilon(r) = \frac{2}{r^2 - 1}\left(2\frac{F_K^2 M_K^2}{F_\pi^2 M_\pi^2} - r - 1\right) \tag{49}$$

$$\eta(r) = \frac{1}{r - 1}\left(\frac{F_K^2}{F_\pi^2} - 1\right). \tag{50}$$

Using more recent results for scattering length from the NA48/2 collaboration

$$a_0^0 = 0.2196 \pm 0.00024 \tag{51}$$

$$\begin{aligned} a_0^2 = & -0.0444 \pm 0.0008 + 0.236(a_0^0 - 0.22) - \\ & - 0.61(a_0^0 - 0.22)^2 - 9.9(a_0^0 - 0.22)^3, \end{aligned} \tag{52}$$

We may compare it with Colangelo et al. [2001] and Descotes et al. [2002], which used data from E865

	a_0^0	a_0^2
Descotes et al. [2002]	0.228 ± 0.012	-0.0382 ± 0.0038
Colangelo et al. [2001]	0.22 ± 0.005	-0.0444 ± 0.001

Table: Comparison of scattering lengths

Representation matching can be found in "$\pi\pi$ scattering" Colangelo et al. [2001]

Using solutions to the Roy equations

$\downarrow$

Phenomenological representation (p_i's)

$\downarrow$

Transform them into chiral representation (c_i's)

$\downarrow$

Transform them into representation introduced by Bijnens ($\bar{b}_i$'s)

$\downarrow$

Get $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$

$$\begin{aligned} A_{\pi\pi}(s,t,u) =& 16\pi a_0^2 + \frac{4\pi}{3M_\pi^2}\left(2a_0^0 - 5a_0^2\right) s + P(s,t,u) + 32\pi[\frac{1}{3}\overline{W}^0(s) + \\ & \frac{3}{2}(s-u)\overline{W}^1(t) + \frac{3}{2}(s-t)\overline{W}^1(u) + \frac{1}{2}\overline{W}^2(u) + \frac{1}{2}\overline{W}^2(u) - \\ & \frac{1}{3}\overline{W}^2(s)] + O(p^8), \end{aligned} \tag{53}$$

where $P(s,t,u)$ is a crossing symmetry polynomial

$$P(s,t,u) = p_1 + p_2 s + p_3 s^2 + p_4 (t-u)^2 + p_5 s^3 + p_6 s (t-u)^2$$

$$p_1 = -128\pi M_\pi^4 \left(\overline{I}_0^1 + \overline{I}_0^2 + 2M_\pi^2 \overline{I}_1^1 + 2M_\pi^2 \overline{I}_1^2 + 8M_\pi^4 \overline{I}_2^2\right)$$

$$p_2 = \frac{-64\pi M_\pi^2}{3}\left(2\overline{I}_0^0 - 6\overline{I}_0^1 - \overline{I}_0^2 - 15M_{/pi}^2 \overline{I}_1^1 - 3M_\pi^2 \overline{I}_1^2 - 36M_\pi^4 \overline{I}_2^2 + 6M_\pi^2 H\right)$$

$$p_3 = \frac{8\pi}{3}\left(4\overline{I}_0^0 - 9\overline{I}_0^1 - \overline{I}_0^2 - 16M_\pi^2 \overline{I}_1^0 - 42M_\pi^2 \overline{I}_1^1 + 22M_\pi^2 \overline{I}_1^2 - 72M_\pi^4 \overline{I}_2^2 + 24M_\pi^2 H\right)$$

$$p_4 = 8\pi\left(4\overline{I}_0^1 + \overline{I}_0^2 + 2M_\pi^2 \overline{I}_1^1 + 2M_\pi^2 \overline{I}_1^2 - 24M_\pi^4 \overline{I}_2^2\right)$$

$$p_5 = \frac{4\pi}{3}\left(8\overline{I}_1^0 + 9\overline{I}_1^1 - 11\overline{I}_1^2 - 32M_\pi^2 \overline{I}_2^0 + 44M_\pi^2 \overline{I}_2^2 - 6H\right)$$

$$p_6 = 4\pi\left(\overline{I}_1^1 - 3\overline{I}_1^2 + 12M_\pi^2 \overline{I}_2^2 + 2H\right),$$

with $\bar{I}^I_n$ and H defined as

$$\bar{I}^I_n = \sum_{l=0}^{\infty} \frac{2l+1}{\pi} \int_{4M_\pi^2}^{\infty} \frac{\mathrm{Im}\, t^I_l(s)}{s^{n+2}(s-4M_\pi^2)} \tag{54}$$

$$H = \sum_{l=2}^{\infty} \frac{(2l+1)l(l+1)}{\pi} \int_{4M_\pi^2}^{\infty} \frac{2\mathrm{Im}\, t^0_l(s) + 4\mathrm{Im}\, t^2_l(s)}{9s^3(s-4M_\pi^2)}. \tag{55}$$

The amplitude can be expressed as

$$t^I_l(s) = \frac{1}{\sigma(s)} e^{i\delta^I_l(s)} \sin(\delta^I_l(s)), \tag{56}$$

where $\delta(s)$ is real. We can then use Schenk parametrization

$$\tan(\delta^I_l) = \sqrt{1-\frac{4M_\pi^2}{s}}\, q^{2l} \left(A^I_l + B^I_l q^2 + C^I_l q^4 + D^I_l q^6 \right) \frac{4M_\pi^2 - s^I_l}{s - s^I_l}. \tag{57}$$

$A^I_l, B^I_l, C^I_l, D^I_l, s^I_l$ are called Schenk parameters.

$$\begin{aligned}
c_1 &= 16\pi a_0^2 + p_1 + O(p^8) \\
c_2 &= \frac{4\pi}{3M_\pi^2}(2a_0^0 - 5a_0^2) + p_2 + O(p^6) \\
c_3 &= p_3 + O(p^4) \\
c_4 &= p_4 + O(p^4) \\
c_5 &= p_5 + O(p^2) \\
c_6 &= p_6 + O(p^2)
\end{aligned} \tag{58}$$

$$
\begin{aligned}
c_1 =& -\frac{M_\pi^2}{F_\pi^2}[1+\xi\left(-\bar{b}_1-\frac{68}{315}\right)\\
&+\xi^2\left(-\frac{8\bar{b}_1}{105}-\frac{32\bar{b}_2}{63}-\frac{464\bar{b}_3}{315}-\frac{3824\bar{b}_4}{315}+\frac{601\pi^2}{945}-\frac{17947}{2835}\right)]\\
c_2 =& \frac{1}{F_\pi^2}[1+\xi\left(\bar{b}_2-\frac{323}{1260}\right)+\\
&\xi^2\left(-\frac{11\bar{b}_1}{70}-\frac{211\bar{b}_2}{315}-\frac{628\bar{b}_3}{315}-\frac{5164\bar{b}_4}{315}-\frac{3977}{630}+\frac{5237\pi^2}{7560}\right)]\\
c_3 =& \frac{1}{NF_\pi^4}\left(\bar{b}_3+\frac{1}{42}+\xi\left(\frac{18\bar{b}_1}{35}+\frac{59\bar{b}_2}{105}+\frac{731\bar{b}_3}{315}+\frac{3601\bar{b}_4}{315}-\frac{5387\pi^2}{15120}-\frac{19121}{7560}\right)\right]\\
c_4 =& \frac{1}{NF_\pi^4}\left(\bar{b}_4-\frac{31}{2520}+\xi\left(-\frac{43\bar{b}_1}{420}-\frac{8\bar{b}_2}{63}+\frac{23\bar{b}_3}{63}+\frac{997\bar{b}_4}{315}+\frac{467\pi^2}{7560}-\frac{63829}{45360}\right)\right]\\
c_5 =& \frac{1}{N^2F_\pi^6}\left(\frac{137}{1680\xi}+\frac{\bar{b}_1}{16}+\frac{379\bar{b}_2}{1680}-\frac{25\bar{b}_3}{28}-\frac{731\bar{b}_4}{180}+\bar{b}_5+\frac{269\pi^2}{15120}+\frac{61673}{18144}\right)\\
c_6 =& \frac{1}{N^2F_\pi^6}\left(-\frac{31}{1680\xi}+\frac{\bar{b}_1}{112}-\frac{47\bar{b}_2}{1680}-\frac{65\bar{b}_3}{252}-\frac{547\bar{b}_4}{420}+\bar{b}_6+\frac{\pi^2}{15120}+\frac{44287}{90720}\right),
\end{aligned}
\tag{59}
$$

To finally receive

$$\alpha_{\pi\pi} = 1 + \xi(3\overline{b}_1 + 4\overline{b}_2 + 4\overline{b}_3 - 4\overline{b}_4) - \frac{11}{36}\pi^2\xi^2 - \frac{152}{9}\xi^2 \quad (60)$$

$$\beta_{\pi\pi} = 1 + \xi(\overline{b}_2 + 4\overline{b}_3 - 4\overline{b}_4) + 4\xi^2(3\overline{b}_5 - \overline{b}_6) - \frac{13}{72}\pi^2\xi^2 + \frac{152}{9}\xi^2, \quad (61)$$

where $\xi = \left(\frac{M_\pi}{4\pi F_\pi}\right)^2$ and $N = 16\pi^2$.

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$\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from $\pi\pi$ scattering - phase shifts

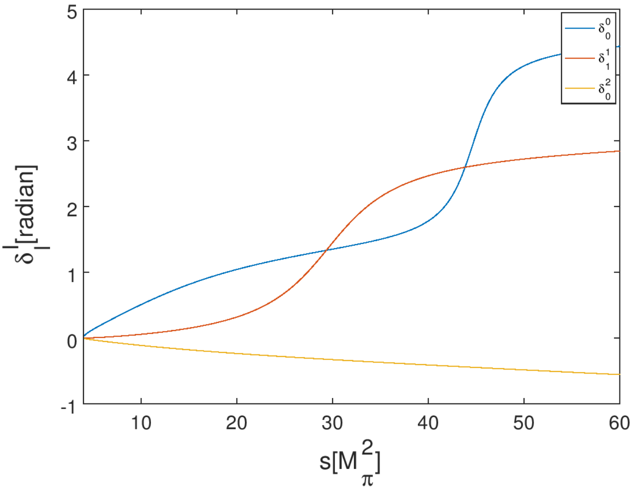


Figure: Dependence of phase shifts on s

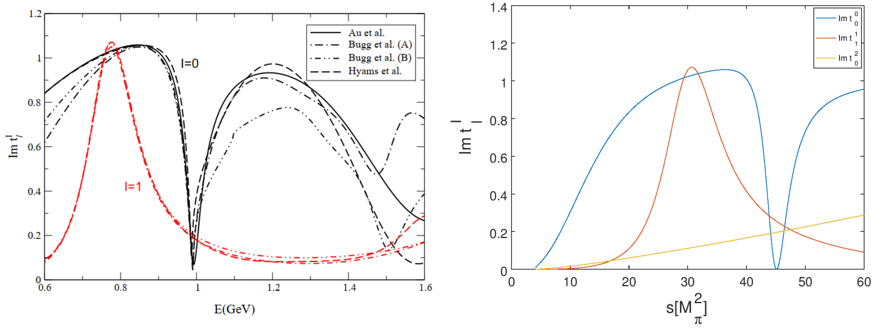


Figure: Comparison of imaginary parts of the amplitudes

	Descotes et al. [2002]	Colangelo et al. [2001]	This work
$\overline{b}_1$	-2.08 ± 6.12	-12.4 ± 1.6	-11.94 ± 1.76
$\overline{b}_2$	9.35 ± 1.43	11.8 ± 0.6	11.41 ± 0.69
$\overline{b}_3$	-0.38 ± 0.03	-0.33 ± 0.07	-0.37 ± 0.01
$\overline{b}_4$	0.716 ± 0.008	0.74 ± 0.01	0.72 ± 0.01
$\overline{b}_5$	3.21 ± 0.25	3.58 ± 0.37	3.26 ± 0.18
$\overline{b}_6$	2.23 ± 0.07	2.35 ± 0.02	2.28 ± 0.04

Table: Comparison of $\overline{b}_i$'s

$\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from $\pi\pi$ scattering

We obtained the subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$. The results were

$$\begin{aligned} \alpha_{\pi\pi} &= 1.08 \pm 0.08 \\ \beta_{\pi\pi} &= 1.11 \pm 0.01 \end{aligned} \tag{62}$$

	$\alpha_{\pi\pi}$	$\beta_{\pi\pi}$
Stern et al.	1.381 ± 0.242	1.081 ± 0.023
Colangelo et al.	1.08 ± 0.07	1.12 ± 0.01
Our results	1.08 ± 0.08	1.11 ± 0.01

Table: Comparison of subthreshold parameters

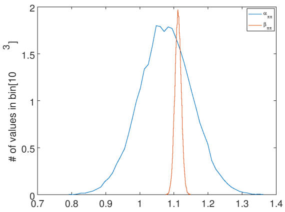


Figure: Histogram of $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ calculated from Roy equations

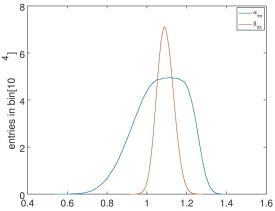


Figure: Histogram for $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ generated from χPT

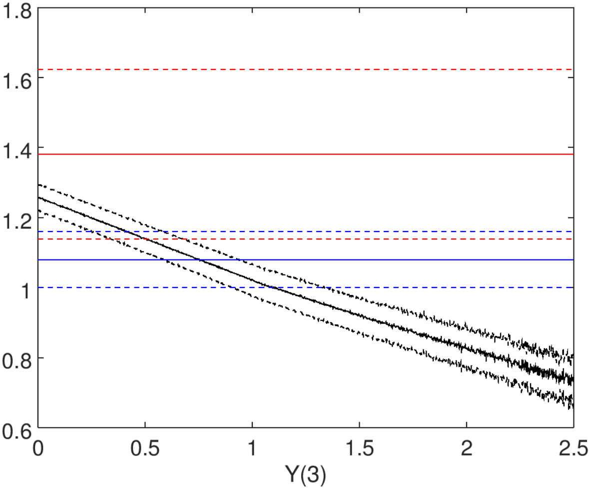
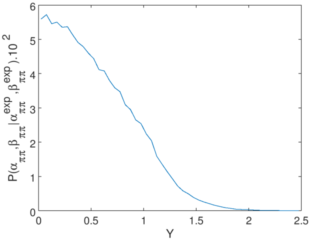


Figure: Comparison of $\alpha_{\pi\pi}$. Theoretical predictions from χPT are black, results from our work in blue and results from Stern[2002] in red

Then we used Monte Carlo with 10^6 entries to simulate χPT parameters X,Y and Z, calculated the subthreshold parameters and applied Bayes's theorem, resulting in following probability distributions

Stern [2002]



Our work

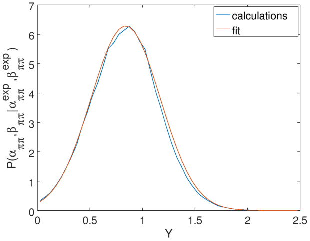


Table: Comparison of probability distributions between our work and Stern et al.

Result for $Y(3)$:

$$Y(3) = 0.84 \pm 0.33 \qquad (63)$$

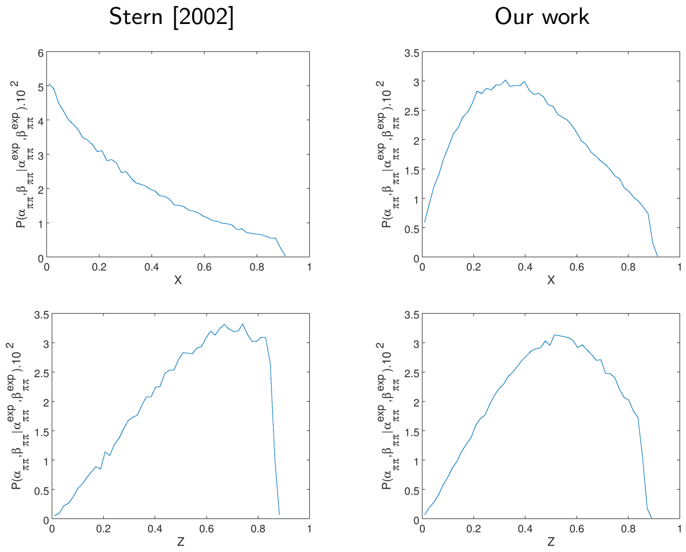


Table: Comparison of probability distributions between our work and Stern et al.

Comparison of results for spontaneous symmetry breaking parameters $X(3)$, $Y(3)$ and $Z(3)$

phenomenology		X(3)	Y(3)	Z(3)
Bijnens and Ecker [2014]	main fit	0.63	1.07	0.59
Bijnens and Ecker [2014]	free fit	0.45	0.94	0.48
Amoros et al. [2001]	fit 10	0.66	0.74	0.89
Kolesár and Novotny [2018]	$\eta \to 3\pi$		1.44 ± 0.32	<0.78
lattice				
Bernard et al. [2012]	RBC/UKQCD	0.38 ± 0.05	0.71 ± 0.10	0.54 ± 0.06
Ecker et al. [2014]	RBC/UKQCD			0.91 ± 0.08
Aoki et al. [2020]	MILC 09A	0.61 ± 0.06	0.84 ± 0.11	0.72 ± 0.06

Table: Comparison of $X(3)$, $Y(3)$ and $Z(3)$

Summary

- We calculated π-π phase shifts
- We compared $\overline{b}_i$ with results from literature
- We have calculated the subthreshold parameters $\alpha_{\pi\pi}$ and $\beta_{\pi\pi}$ from Roy equations:

$$\begin{aligned} \alpha_{\pi\pi} &= 1.08 \pm 0.08 \\ \beta_{\pi\pi} &= 1.11 \pm 0.01 \end{aligned} \tag{64}$$

- We have used the resummed approach with Bayesian approach to produce probability distributions for χPT LO parameters X, Y, Z
- Significant shift in probability distributions $\rightarrow$ more consistent with theoretical expectations
- Result for Y(3):

$$Y(3) = 0.84 \pm 0.33 \tag{65}$$

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